String attractors, or how to capture text combinatorics

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Lempel-Ziv 77

- Loseless text compression algorithm
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Example:

abracadabra banana

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Example:

a.bracadabra banana

(0,0,a)

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Example:

a.b.racadabra banana

(0,0,a)(0,0,b)

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Example:

a.b.r.acadabra banana

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Example:

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(0,0,a)(0,0,b)(0,0,r)(1,1,c)(1,1,d)

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Example:

a.b.r.ac.ad.abra .banana

(0,0,a)(0,0,b)(0,0,r)(1,1,c)(1,1,d)(1,4,)

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Example:

a.b.r.ac.ad.abra .ba.nana

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a.b.r.ac.ad.abra .ba.n.ana

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a.b.r.ac.ad.abra .ba.n.ana

(0,0,a)(0,0,b)(0,0,r)(1,1,c)(1,1,d)(1,4,)(2,1,a)(0,0,n)(14,3,EOF)

Burrows-Wheeler Transform

- Another loseless text compression algorithm
- More efficient on shorter text

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Example:

b a n a n a	a b a n a **n**
a n a n a b	a n a b a **n**
n a n a b a	a n a n a **b**
a n a b a n	b a n a n **a**
n a b a n a	n a b a n **a**
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banana $\rightsquigarrow$ nnbaaa

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$$\text{banana} \rightsquigarrow \text{nnbaaa} \rightsquigarrow \text{(n,2)(b,1)(a,3)}$$

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n a b a n a	n a b a n **a**
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$$\text{banana} \rightsquigarrow \text{nnbaaa} \rightsquigarrow (\text{n},2)(\text{b},1)(\text{a},3)$$

Can we measure the repetitiveness of a text?

String attractors

Definition (Kempa & Prezza 2017)

A *string attractor* of a text of length n is a set $\Gamma \subseteq \{1, \ldots, n\}$ of positions such that every substring (factor) of the text has an occurrence crossing a position in Γ.

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Definition

$\gamma^*(t) =$ minimal size of a string attractor of t.

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a.b.r.ac.ad.abra .ba.n.ana $\rightsquigarrow$ abracadabra_banana

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If

- $lz(t)$ *is the number of phrases in the LZ77 compression of* t,

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1. $\gamma^*(t) \leq lz(t)$,
2. $\gamma^*(t) \leq bwt(t)$,
3. *$\gamma^*(t)$ also gives upper bounds on $lz(t)$ and $bwt(t)$.*

Complexity

Proposition (Kempa & Prezza)

Let $k \geq 3$ and $n \in \mathbb{N}$.
Deciding whether a given text admits a k-attractor of size at most n is NP-complete.

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Complexity

Proposition (Kempa & Prezza, Fuchs & Whittington)

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Can we use combinatorial ideas to find optimal attractors for particular examples?

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3. *is a string attractor for* tp *for any prefix* p *of* $ttttt\cdots$.

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Corollary

1. $\gamma^*(t^R) = \gamma^*(t)$*;*
2. $\gamma^*(ts) \leq \gamma^*(t) + \gamma^*(s) + 1$*;*
3. $\gamma^*(tp) \leq \gamma^*(t) + 1$ *for every prefix* p *of* $tttt\cdots$*.*

Monotonicity

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  No, example:
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  No, there are infinitely many texts t s.t. $\gamma^*(tt) \sim \frac{1}{2}\gamma^*(t)$.
  (Mantaci et al.)

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- $s_x(n) \leq n$ for all n;
- $s_x(n) \leq \mathrm{lz}(x[1, n]) \in O\left(\frac{n}{\log_{\#A}(n)}\right)$.

The bounded case

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Are there aperiodic sequences with bounded profile function?

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01

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01101001

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0, 01, 010, 01001, 01001010, 0100101001001, ... $\rightarrow$
$s_x(n) \leq 2$

The bounded case (2)

In both examples, every factor appears with linearly bounded gaps.

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Proposition

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- *If every factor appears with linearly bounded gaps, then s_x is bounded. (Schaeffer & Shallit)*
- *If s_x is bounded, then either x is eventually periodic, or the number of different factors grows linearly. (Restivo, Romana & Sciortino)*

Fibonacci: first ideas

Construction of the string attractors:

$$0100101001001\cdots$$

String attractors:

Fibonacci: first ideas

Construction of the string attractors:

$$\underline{0}1001010010001\cdots$$

String attractors:

$$\{1\}$$

Fibonacci: first ideas

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String attractors:

$$\{1\}, \{1, 2\}$$

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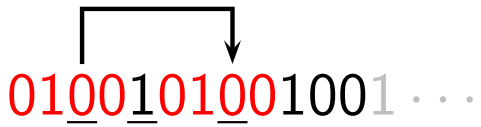

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Main result

Let

$$f_0 = 0, \quad f_1 = 01, \quad f_{n+2} = f_{n+1} f_n$$

and $F_n = \mathrm{len}(f_n)$.

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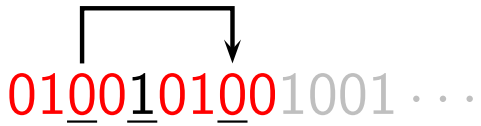

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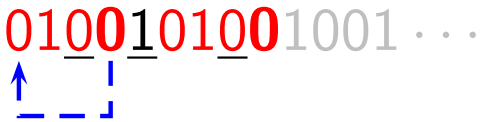

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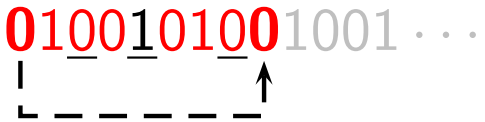

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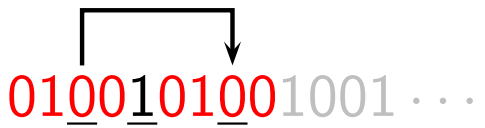

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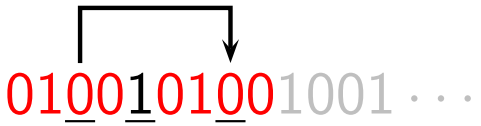

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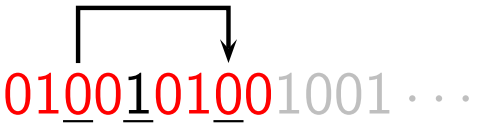

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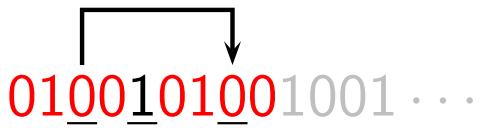

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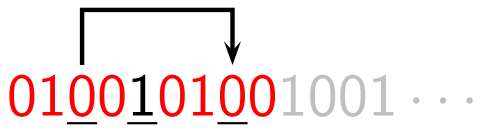

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Tribonacci sequence

$$t_0 = 0, \quad t_1 = 01, \quad t_2 = 0102, \quad t_{n+3} = t_{n+2}t_{n+1}t_n$$

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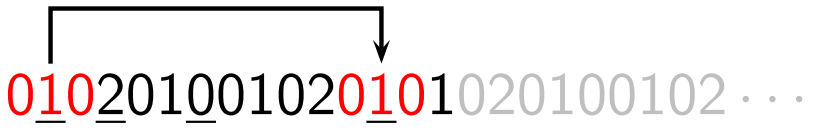

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Proposition (Cassaigne et al.)

Every prefix has a s.a. made of (at most) 3 consecutive Tribonacci numbers.

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Proposition (Cassaigne et al.)

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More generally, if $x = \lim_{i\to\infty} u_i$ *with*

$$u_i = \begin{cases} u_{i-1}u_{i-2}\cdots u_0\ i & \text{if } i < k, \\ u_{i-1}u_{i-2}\cdots u_{i-k} & \text{if } i \geq k, \end{cases}$$

then every prefix of x *has a s.a. made of (at most)* k *consecutive elements of* $\{\mathrm{len}(u_i) : i \in \mathbb{N}\}$.

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Consider the substitutions

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Proposition (Cassaigne et al.)

If $x = \lim_{i\to\infty} \varphi_{1^k}^i(0)$, then every prefix of x has a s.a. made of (at most) k consecutive elements of $\{\mathrm{len}(\varphi_{1^k}^i(0)) : i \in \mathbb{N}\}$.

Generalizing

For some parameters $c_0, \ldots, c_{k-1} \in \mathbb{N}$ with $c_0, c_{k-1} \geq 1$, we define

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Does every prefix of $\lim_i \varphi^i_{c_0 \cdots c_{k-1}}(0)$ have a s.a. made of (at most) k consecutive U_n's?

Answer

Proposition (G., Romana & Stipulanti)

Let $x = \lim_i \varphi^i_{c_0 \cdots c_{k-1}}(0)$

- *If there exists* j *such that* $c_j \cdots c_{k-2}(c_{k-1}-1)c_0 \cdots c_{j-1} > c_0 \cdots c_{k-2}(c_{k-1}-1)$, *then there are prefixes of* x *having no s.a. included in* $\{U_i : i \in \mathbb{N}\}$.
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Proposition (Dvoraková & Moravcová)

If $c_j \cdots c_{k-2}(c_{k-1}-1)c_0 \cdots c_{j-1} \leq c_0 \cdots c_{k-2}(c_{k-1}-1)$ *for every* j, *then every prefix of* x *has a s.a. of size (at most)* k.

Other string attractors

Construction of the string attractors:

$$0100101001001\cdots$$

String attractors:

Other string attractors

Construction of the string attractors:

$$\underline{0}100101001001\cdots$$

String attractors:

$$\{1\}$$

Other string attractors

Construction of the string attractors:

$$\underline{01}00101001001\cdots$$

String attractors:

$$\{1\}, \{1, 2\}$$

Other string attractors

Construction of the string attractors:

$$\underline{010}0101001001\cdots$$

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Other string attractors

Construction of the string attractors:

$$010\underline{01}0\,1001001\cdots$$

String attractors:

$$\{1\}, \{1,2\}, \{2,3\}, \{4,5\}$$

Other string attractors

Construction of the string attractors:

$$0100\underline{1}101001001\cdots$$

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String attractors:

$$\{1\},\{1,2\},\{2,3\},\{4,5\},\{7,8\},\ldots$$

Other string attractors

Construction of the string attractors:

$$0100101\underline{0}0100\,1\cdots$$

String attractors:

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Alternative construction of the sequence

0

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0

0

Alternative construction of the sequence

0

01

Alternative construction of the sequence

0

010

Alternative construction of the sequence

0

010

010

Alternative construction of the sequence

0

010

0100

Alternative construction of the sequence

0

010

010010

Alternative construction of the sequence

0

010

010010

010010

Alternative construction of the sequence

0

010

010010

0100101

Alternative construction of the sequence

0

010

010010

01001010010

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010

010010

01001010010

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010

010010

01001010010

0100101001010

Alternative construction of the sequence

0

010

010010

01001010010

0100101001001010010

...

Alternative construction of the sequence

0

010

010010

01001010010

0100101001001010010

...

Proposition (Restivo, Romana & Sciortino)

Every prefix has a string attractor made of 2 consecutive positions.

Alternative construction of the sequence

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010010

01001010010

0100101001001010010

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Proposition (Restivo, Romana & Sciortino)

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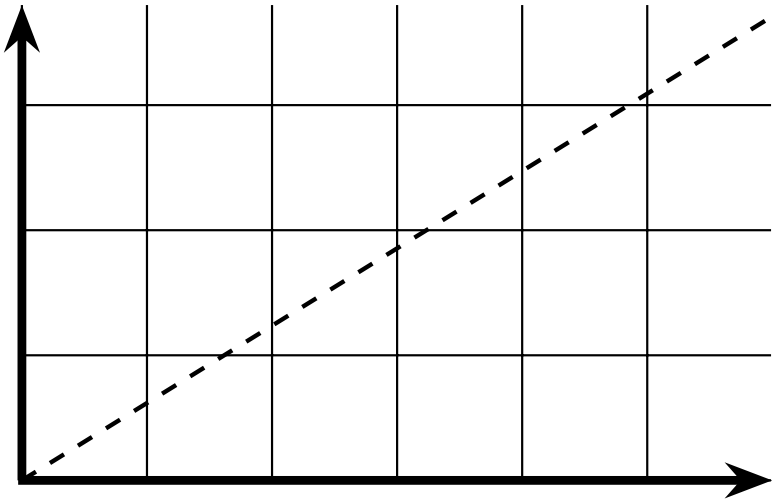
Fibonacci is the *characteristic Sturmian* sequence with directive sequence $010101\cdots$.

Sturmian sequences

Take a line with irrationnal slope,



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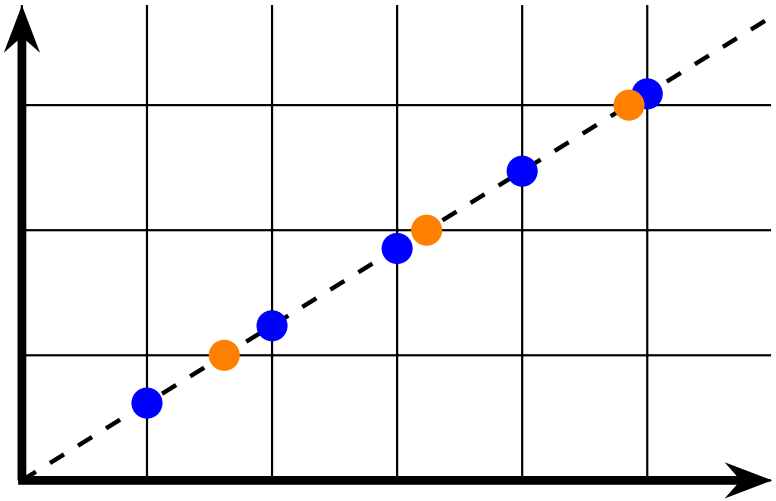

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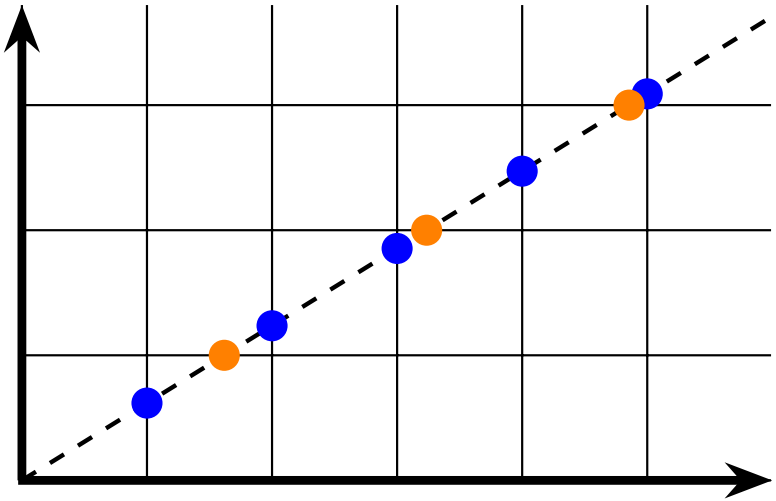

- intersections with the grid (billiard)

Sturmian sequences

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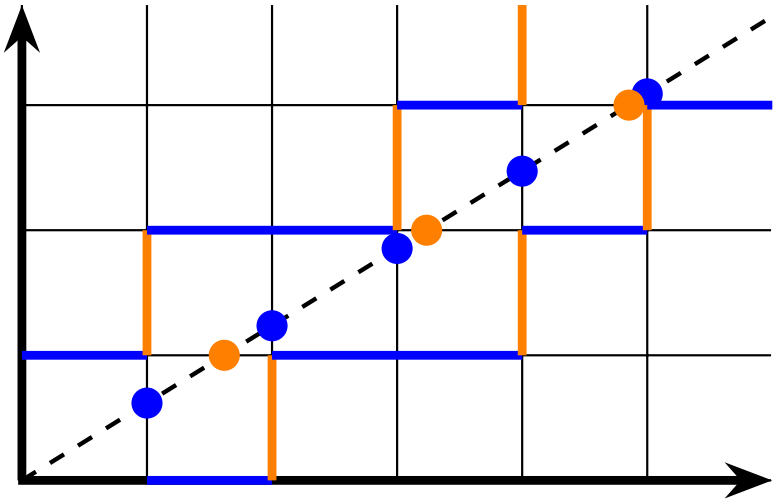

- intersections with the grid (billiard)
- approximation by grid lines

Sturmian sequences

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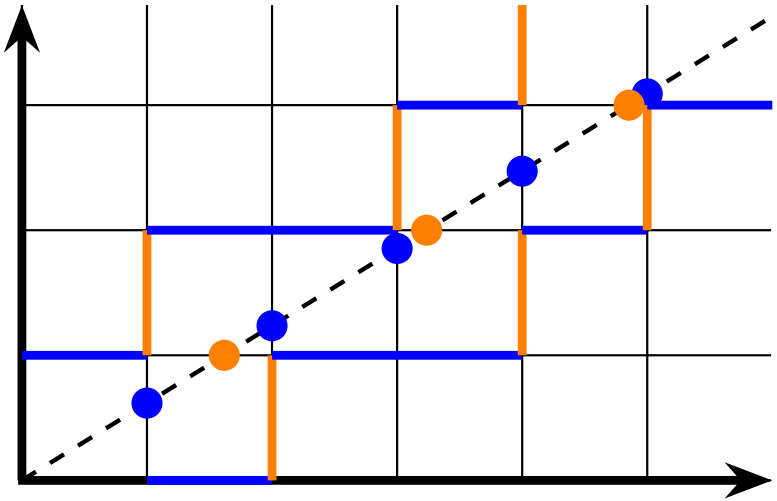

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you get a binary sequence called *Sturmian*.

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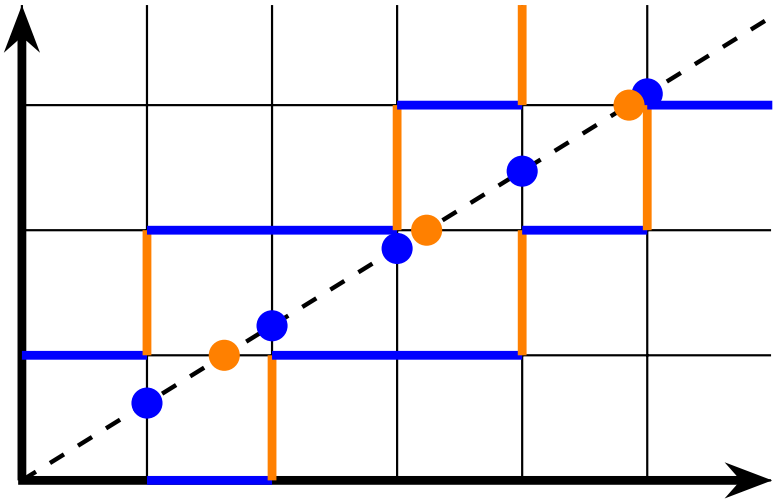

- intersections with the grid (billiard)
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you get a binary sequence called *Sturmian*.

If the intercept is zero, it is *characteristic Sturmian*.

Sturmian sequences and string attractors

Proposition (de Luca)

Characteristic Sturmian sequences are obtained by iterated palindromic closure for a directed sequence with 0's and 1's (infinitely many of each).

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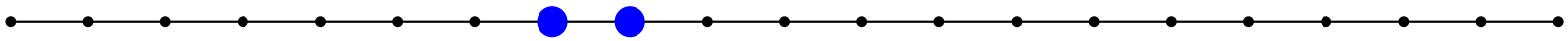
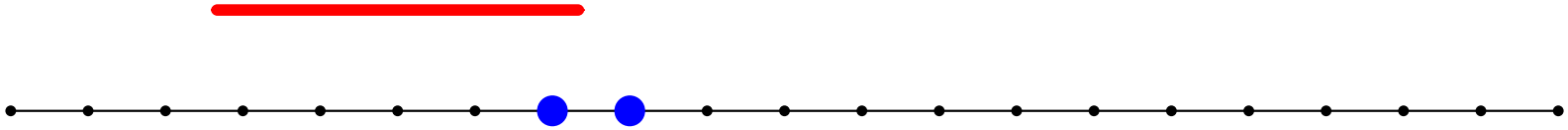
What about the converse?

Sturmian sequences and string attractors (2)



Sturmian sequences and string attractors (2)



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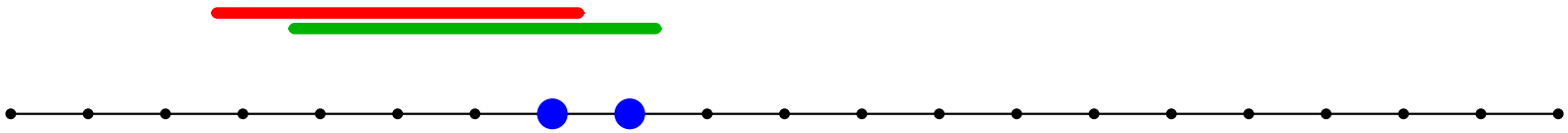

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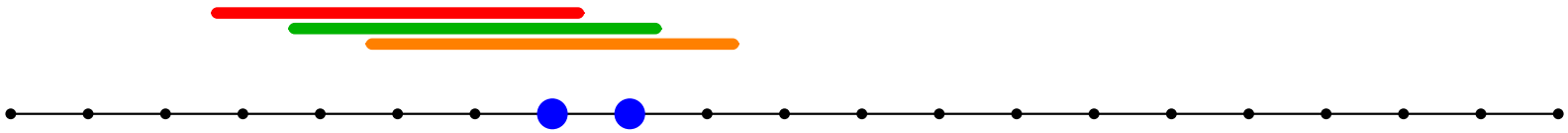

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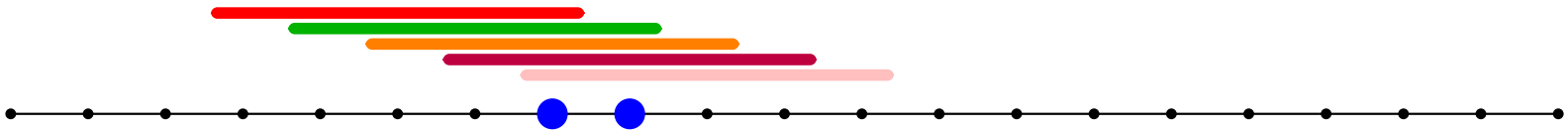

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Proposition (Restivo, Romana & Sciortino)

A sequence is Sturmian if and only if it is aperiodic and infinitely many prefixes have a string attractor made of 2 consecutive positions.

Conclusion and open questions

Summary:

- combinatorial object related to text compression

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Open questions:

- Can we characterize the sequences such that s_x is bounded?
- Can we describe string attractors for other substitutive sequences?
- Can we characterize other well-known families of sequences through string attractors?

The end

Thank you for listening!