Algebraic properties of return words

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Introduction

Context

- X is a shift space indexed by $\mathbb{Z}$
- its alphabet is $\mathcal{A}$
- its langage is $\mathcal{L}(X) = \{w \in \mathcal{A}^* : \exists x \in X, i \in \mathbb{Z} \mid x_{[i,i+|w|)} = w\}$

Return words

$$\cdots aabbaacaabbaaccaabbaaabb \cdots$$

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Definition

A **return word** for w is a word u such that

$$uw \in \mathcal{L}(X) \cap w\mathcal{A}^+ \setminus \mathcal{A}^+ w \mathcal{A}^+.$$

The **set of return words** for w is denoted $\mathcal{R}_X(w)$.

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- we always assume that X is minimal
- $\mathcal{R}_X(\varepsilon) = \mathcal{A}$

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- useful to build particular S-adic representations
- used to study factor occurrences, critical exponents, skew products, ...

First results

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1. *there exists* $w \in \mathcal{L}(X)$ *such that* $\#\mathcal{R}_X(w) = 1$;

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Theorem (Vuillon)

X *is Sturmian if and only if* $\#\mathcal{R}_X(w) = 2$ *for all* $w \in \mathcal{L}(X)$.

Counting return words

For any $w \in \mathcal{L}(X)$,

$$\mathrm{E}_X^L(w) = \{a \in \mathcal{A} : aw \in \mathcal{L}(X)\} \qquad \mathrm{E}_X^R(w) = \{b \in \mathcal{A} : wb \in \mathcal{L}(X)\}$$

$$\mathrm{E}_X(w) = \{(a, b) \in \mathcal{A}^2 : awb \in \mathcal{L}(X)\}$$

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Proposition (based on ideas from Balková, Pelantová, Steiner)

For every $w \in \mathcal{L}(X)$,

$$\#\mathcal{R}_X(w) = \#\mathcal{A} + \sum_{\substack{u \in \mathcal{L}(X) \\ |u|_w = 0}} \underbrace{\#\mathrm{E}_X(u) - \#\mathrm{E}_X^L(u) - \#\mathrm{E}_X^R(u) + 1}_{\text{multiplicity of } u}.$$

Constant number of return words

Proposition (Balková, Pelantová, Steiner)

- *For $m \leq 3$, $\#\mathcal{R}_X(w) = m$ for all $w \in \mathcal{L}(X)$ if and only if all multiplicities are zero (i.e. X is* neutral*) and $\#\mathcal{A} = m$.*

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- *For $m \geq 4$, $\#\mathcal{R}_X(w) = m$ for all $w \in \mathcal{L}(X)$ if ~~and only if~~ all multiplicities are zero (i.e. X is* neutral*) and $\#\mathcal{A} = m$.*

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Embedding in groups

$\mathcal{R}_X(w) \in \mathcal{A}^*$

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$$\mathcal{R}_X(w) \in \mathcal{A}^* \qquad \hookrightarrow \qquad F_{\mathcal{A}}$$

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- $\varphi = \mathrm{id}$ and $G = F_{\mathcal{A}}$
- $\varphi = |\cdot|$ and $G = \mathbb{Z}$, ex: $|aab^{-1}| = 1$
- $\varphi = \mathrm{Ab}$ and $G = \mathbb{Z}^{\#\mathcal{A}}$, ex: $\mathrm{Ab}(aab^{-1}) = (2, -1)$

Return groups

What can we say about the sets $\varphi(\mathcal{R}_X(w))$, $w \in \mathcal{L}(X)$, in G?

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Definition

X has φ-**full return groups** if $\varphi\langle \mathcal{R}_X(w) \rangle = G$ for all $w \in \mathcal{L}(X)$.

Only for long enough words

Lemma

If $w = pus$, then

$$\langle \mathcal{R}_X(w) \rangle \leq p \langle \mathcal{R}_X(u) \rangle p^{-1}.$$

Idea: any return word to w is a concatenation of return words to u (up to conjugacy by p).

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Proposition

If $\varphi\langle\mathcal{R}_X(w)\rangle = G$ for infinitely many $w \in \mathcal{L}(X)$, then X has φ-full return groups.

Idea: use the uniform recurrence

Return groups stabilization

Definition

X has **eventually φ-stable** return groups if there exists a subgroup $H \leq G$ such that $\varphi\langle\mathcal{R}_X(w)\rangle$ is conjugated to H for all $w \in \mathcal{L}_{\geq N}(X)$.

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Remarks:

- We allow conjugacy because of the previous lemma.

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Remarks:

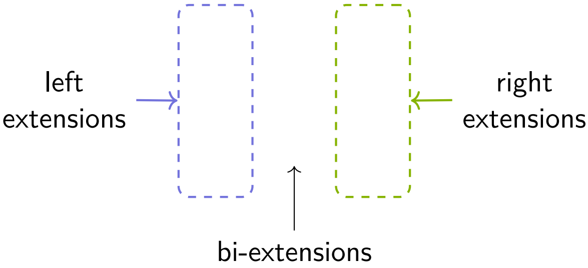
- We allow conjugacy because of the previous lemma.
- If $N = 0$ and φ is onto, then X has φ-full return groups.

Link with existing results

Extension graphs

$$\cdots baa\,ba\,ab\,ba\,abaabaab\,ba\,bbaa \cdots$$

$\mathcal{E}_X(ba)$



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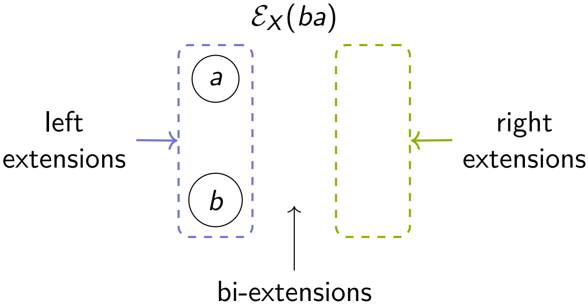

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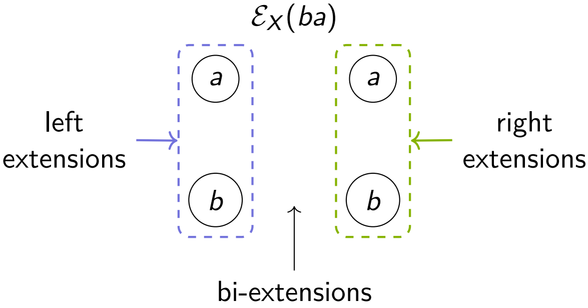

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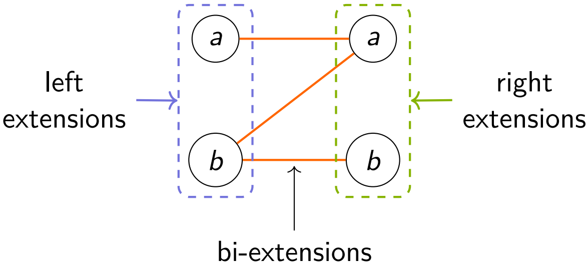

Connected shifts

Definition

X is

- **connected** if $\mathcal{E}_X(w)$ is connected for all $w \in \mathcal{L}(X)$;
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Proposition (Berthé, De Felice, Dolce, Leroy, Perrin, Reutenauer, Rindone)

1. *If X is connected, then it has* id*-full return groups.*
2. *If X is eventually connected, then it has eventually* id*-stable return groups.*

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Also works for (eventually) suffix-connected shifts [Goulet-Ouellet].

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Proposition (Berthé *et al.* ; G., Goulet-Ouellet, Leroy, Stas)

X is dendric if and only if $\mathcal{R}_X(w)$ is a basis of $F_{\mathcal{A}}$ for all $w \in \mathcal{L}(X)$.

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1. *X is totally minimal,*
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3. *X has $|\cdot|$-full return groups.*

Isomorphism invariant

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The following properties are preserved by isomorphism (= topological conjugacy):

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Isomorphism invariant

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The following properties are preserved by isomorphism (= topological conjugacy):

1. *being totally minimal,*
2. *having eventually $|\cdot|$-stable return groups and stabilizing on $k\mathbb{Z}$.*

Welldoc property

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A sequence $x \in \mathcal{A}^{\mathbb{N}}$ has **welldoc** if, for all $w \in \mathcal{L}(x)$ and all $k \geq 2$,

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$$\varphi(\{p : pw \text{ prefix of } x\}) = G.$$

welldoc $= \mathrm{Ab}_2$-welldoc $\wedge$ Ab_3-welldoc $\wedge$ Ab_4-welldoc $\wedge \ldots$

Link with full return groups

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Proposition (Berthé, Goulet-Ouellet, Nyberg Brodda, Perrin, Petersen)

If G is finite, X has φ-full return groups if and only if $x_{[0,+\infty)}$ has φ-welldoc for any $x \in X$.

Sequences with welldoc

Proposition (G., Goulet-Ouellet, Leroy, Stas)

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Examples having welldoc:

- Sturmian and Arnoux-Rauzy sequences [Balková *et al.*]
- (suffix-)connected sequences
- some of the sequences with unimodular S-adic representations [Puzynina, Schavelev; Esponiza]

Tools to study stabilization

Known families

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- eventually (suffix-)connected shifts have eventually φ-stable return groups for all φ
- aperiodic automatic shifts **do not** have eventually φ-stable return groups for all φ such that $|\cdot| = \psi \circ \varphi$ for some ψ [Krawczyk, Müllner]

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If G is LERF, then X has eventually φ-stable return groups if and only if the subgroups $\varphi\langle\mathcal{R}_X(w)\rangle$ are conjugate for infinitely many $w \in \mathcal{L}(X)$.

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Examples of LERF (or subgroup separable) groups:

- finite groups
- free groups
- finitely generated Abelian groups

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Corollary

If G is finite, then X has eventually φ-stable return groups.

Decidability

Proposition

If X is morphic and G is finite, then we can

1. *compute a subgroup H on which the groups $\varphi\langle\mathcal{R}_X(w)\rangle$ stabilize,*
2. *decide whether X has φ-full return groups.*

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Proposition

If X is generated by a primitive substitution that is bifix or a return morphism, then we can

1. *decide whether X has eventually id-stable return groups,*
2. *compute a subgroup H on which the groups $\langle\mathcal{R}_X(w)\rangle$ stabilize,*
3. *decide whether X has id-full return groups.*

Local-global principle

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Proposition

X has Ab-full return groups if and only if it has Ab_k-full return groups for all $k \geq 1$.

Open questions

- Are there other φ for which φ-stabilization of the return groups is a dynamical property?
- Can we characterize the shifts with φ-full return groups?

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Thank you for your attention!